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Additional Appendix to "Availability of Higher Education and Long-Term Economic Growth" (not to publish)

Derivation of $\widehat{\Delta}$

For reference, we repeat the definition of function Ψ_3:

$$\Psi_3(X_{t-1}; \Delta, \Gamma) \equiv \frac{(1-\alpha)X_{t-1} - \Gamma}{(\Delta+\Gamma)^{\alpha}(X_{t-1}-\Gamma)^{1-\alpha}}. \tag{23}$$

Function Ψ_3 is defined for $X_{t-1} \in \left[\frac{\alpha\Delta+\Gamma}{1-\alpha}, \infty\right)$. We now examine the gradient of the Ψ_3 curve at the left end, i.e., at $X_{t-1} = \frac{\alpha\Delta+\Gamma}{1-\alpha}$. By differentiating (23) with respect to X_{t-1}, we obtain

$$\frac{\partial \Psi_3}{\partial X_{t-1}} = \frac{\alpha(1-\alpha)X_{t-1}}{(\Delta+\Gamma)^{\alpha}(X_{t-1}-\Gamma)^{2-\alpha}}. \tag{29}$$

Substituting $X_{t-1} = \frac{\alpha\Delta+\Gamma}{1-\alpha}$ into (29) yields

$$\frac{\overline{\Delta}(\alpha\Delta+\Gamma)}{(\Delta+\Gamma)^2}, \tag{30}$$

where $\overline{\Delta} \equiv \alpha^{\alpha-1}(1-\alpha)^{2-\alpha}$ is defined in (25).

Our concern is whether expression (30) is greater than unity or not when the left end of the Ψ_3 curve is exactly on the 45-degree line. That is, from (26), we consider the case in which Γ is given by

$$\Gamma = \overline{\Gamma}(\Delta) \equiv (\Delta/2)\left(-(1+\alpha) + (1-\alpha)\sqrt{S(\Delta)}\right), \tag{31}$$

where

$$S(\Delta) \equiv 1 + \frac{4}{\Delta}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}. \tag{32}$$

Note that, using (31) and (32), $\alpha\Delta+\Gamma = (\Delta/2)(1-\alpha)(\sqrt{S(\Delta)}-1)$ and $\Delta+\Gamma = (\Delta/2)(1-\alpha)(\sqrt{S(\Delta)}+1)$. In addition, from $\overline{\Delta} \equiv \alpha^{\alpha-1}(1-\alpha)^{2-\alpha} = (\alpha/(1-\alpha))^{\alpha}\alpha^{-1}(1-\alpha)^2$,

$$S(\Delta) - 1 = \frac{4}{\Delta}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} = \frac{4\alpha\overline{\Delta}}{(1-\alpha)^2\Delta}. \tag{33}$$

Therefore, (30) can be simplified to

$$\begin{aligned}\frac{\partial\Psi_3}{\partial X_{t-1}} =& \frac{2\overline{\Delta}}{(1-\alpha)\Delta}\frac{\sqrt{S(\Delta)}-1}{(\sqrt{S(\Delta)}+1)^2}\\ =& \frac{2\overline{\Delta}}{(1-\alpha)\Delta}\frac{(\sqrt{S(\Delta)}-1)^3}{(S(\Delta)-1)^2}\\ =& \frac{(1-\alpha)^3(\sqrt{S(\Delta)}-1)^3\Delta}{8\alpha^2\overline{\Delta}}. \end{aligned} \tag{34}$$

Define $x \equiv (\overline{\Delta}/\Delta)^{1/3} > 0$. Then condition $\partial\Psi_3/\partial X_{t-1} > 1$, where the left hand side is given by (34), can be written as

$$\begin{aligned}&\frac{(1-\alpha)^3(\sqrt{S(\Delta)}-1)^3}{8\alpha^2}x^{-3} > 1\\ \Leftrightarrow\ & \sqrt{S(\Delta)} > 2\alpha^{2/3}(1-\alpha)^{-1}x+1\\ \Leftrightarrow\ & S(\Delta) > 4\alpha^{4/3}(1-\alpha)^{-2}x^2 + 4\alpha^{2/3}(1-\alpha)^{-1}x+1. \end{aligned} \tag{35}$$

Substituting (33) into condition (35) and dividing both hands by $4\alpha(1-\alpha)^{-2}x$ gives

$$x^2 - \alpha^{1/3}x - \alpha^{-1/3}(1-\alpha) > 0. \tag{36}$$

The LHS of (36) represents a convex quadratic curve and, from $\alpha \in (0,1)$, its value is negative when $x \to 0$ (recall that x is positive). Therefore, using the quadratic formula, (36) is equivalent to

$$x > \alpha^{1/3}(1/2 + \sqrt{\alpha^{-1}-3/4}). \tag{37}$$

Substituting $x \equiv (\overline{\Delta}/\Delta)^{1/3}$ into (37) shows that condition (37) is equivalent to

$$\Delta < \left(3/2 - \alpha + \sqrt{\alpha^{-1}-3/4}\right)^{-1}\overline{\Delta} \equiv \widehat{\Delta}.$$

Proof of Proposition 1

The Ψ_3 curve becomes tangential to the 45-degree line at X if and only if the (Γ, X) pair is a solution to the simultaneous equations:

$$\Psi_3(X;\Delta,\Gamma) = X \quad \text{and} \quad \Psi_3'(X;\Delta,\Gamma) = 1. \tag{38}$$

Substituting (23) and (29) into (38) and then dividing each side of the second equation by the corresponding side of the first equation, we have

$$f(X;\Gamma) \equiv (1-\alpha)^2X^2 - (2-\alpha)X\Gamma + \Gamma^2 = 0. \tag{39}$$

Because $f(\Gamma/(1-\alpha);\Gamma) = -\alpha\Gamma^2/(1-\alpha) < 0$ and because function f is a convex quadratic function, the solution to equation (39) such that $X \geq \frac{\alpha\Delta+\Gamma}{1-\alpha} \geq \Gamma/(1-\alpha)$ is unique.[29]

Denote this solution by $X = \xi_1(\alpha)\Gamma$, where $\xi_1(\alpha)$ is a positive constant that depends only on α. Using this result, we can eliminate X from the equation $\Psi_3'(X;\Delta,\Gamma) = 1$ to get

$$\xi_2(\alpha)\Gamma^{-\frac{1-\alpha}{\alpha}} - \Delta = \Gamma, \tag{40}$$

where $\xi_2(\alpha)$ is another positive constant that also depends only on α. When viewed as a function of Γ, the left hand side is monotonically decreasing and ranges between $-\Delta$ and ∞ while the right hand side is simply a 45-degree line. Thus, there is a unique positive value of Γ that satisfies equation (40). For each $\Delta < \widehat{\Delta}$, this solution gives $\widehat{\Gamma}(\Delta)$ since it and $X = \xi_1(\alpha)\Gamma$ satisfy the simultaneous equations in (38). By totally differentiating both sides of equation (40), keeping α constant, we get

$$\widehat{\Gamma}'(\Delta) = \frac{d\Gamma}{d\Delta} = -\frac{1}{1+\xi_2(\alpha)((1-\alpha)/\alpha)\Gamma^{-1/\alpha}} \in (-1,0).$$

The following confirms that $\lim_{\Delta\to\widehat{\Delta}} \widehat{\Gamma}(\Delta) = \overline{\Gamma}(\widehat{\Delta})$. When $\Delta = \widehat{\Delta}$ and $\Gamma = \overline{\Gamma}(\widehat{\Delta})$, the left end of the Ψ_3 curve (i.e. at $X = \frac{\alpha\Delta+\Gamma}{1-\alpha}$) is exactly on the 45-degree curve and its gradient is unity.[30] This means that $(X,\Gamma) = (\frac{\alpha\Delta+\Gamma}{1-\alpha}, \overline{\Gamma}(\widehat{\Delta}))$ is a solution to (38) when $\Delta = \widehat{\Delta}$. Since the solution to (38) changes continuously with respect to Δ in the neighborhood of $\widehat{\Delta}$, the value of $\widehat{\Gamma}(\Delta)$, which is a solution to (38) for the case of $\Delta < \widehat{\Delta}$, approaches $\overline{\Gamma}(\widehat{\Delta})$ as $\Delta \to \widehat{\Delta}$.

[29]Function Ψ_3 is defined only for $X \geq \frac{\alpha\Delta+\Gamma}{1-\alpha}$.

[30]This follows from the definitions of $\overline{\Gamma}(\Delta)$ and $\widehat{\Delta}$ in (26) and (28).

Proof of Propositions 2 and 3

We prove the propositions by constructing four lemmas in turn. The first one shows that as long as the economy stays in the AK regime, X_t converges to $X^{**}(\Delta)$.[31]

Lemma 1. *Suppose that there exists some $T \geq 0$ such that X_t follows dynamics $X_{t+1} = \Psi_2(X_t; \Delta)$ for all $t \geq T$. Then X_t converges to $X^{**}(\Delta)$.*

Proof. Function $\Psi_2(X_t; \Delta)$ is defined by (22). Let us define a new function Φ by

$$\Phi(X; \Delta) \equiv \Psi_2(\Psi_2(X; \Delta); \Delta) = \left(\frac{\alpha}{1-\alpha}\right)^{\alpha} \left[1 - \frac{\left(\frac{\alpha}{1-\alpha}\right)^{\alpha}}{X + \Delta + \left(\frac{\alpha}{1-\alpha}\right)^{\alpha}}\right]. \tag{41}$$

It is easy to confirm that $\Phi(X; \Delta)$ is both strictly increasing and strictly concave on domain $X > 0$, with $\Phi(0; \Delta) > 0$ and $\lim_{X \to \infty} \Phi'(X; \Delta) = 0$. Also, note that $X^{**}(\Delta)$ must be a fixed point of $\Phi(X; \Delta)$ since it is a fixed point of $\Psi_2(X; \Delta)$. These facts are sufficient to conclude that $\Phi(X; \Delta) > X$ for $0 < X < X^{**}(\Delta)$ and $\Phi(X; \Delta) < X$ for $X > X^{**}(\Delta)$ and that the dynamics $X_{t+2} = \Phi(X_t; \Delta)$ is globally stable. Thus, both the odd-numbered and even-numbered elements in the sequence $\{X_t\}$ converge to $X^{**}(\Delta)$, which implies that the sequence $\{X_t\}$ itself converges to $X^{**}(\Delta)$. □

Using this lemma, we can show that there is an interval from which X_t always converges to $X^{**}(\Delta)$.

Lemma 2. *Suppose that $\Delta \leq \overline{\Delta}$ and $\Gamma \geq \overline{\Gamma}(\Delta)$. Define intervals $J_1(\Delta)$ and $J_2(\Delta, \Gamma)$ by*

$$\begin{aligned} J_1(\Delta) &= \left[\frac{\alpha\Delta}{1-\alpha}, \Psi_2\left(\frac{\alpha\Delta}{1-\alpha}; \Delta\right)\right], \\ J_2(\Delta, \Gamma) &= \left[\Psi_2\left(\frac{\alpha(\Delta+\Gamma)}{1-\alpha}; \Delta\right), \frac{\alpha(\Delta+\Gamma)}{1-\alpha}\right]. \end{aligned}$$

Then,

[31] $X^{**}(\Delta) \equiv (\alpha\Delta + \overline{\Gamma}(\Delta))/(1-\alpha)$ is the unique solution to $\Psi_2(X; \Delta) = X$.

a. Either $\Psi_2(\alpha\Delta/(1-\alpha);\Delta) < (\alpha\Delta+\Gamma)/(1-\alpha)$ *or* $\Psi_2((\alpha\Delta+\Gamma)/(1-\alpha);\Delta) > \alpha\Delta/(1-\alpha)$ *holds. (This means either* $J_1(\Delta)$ *or* $J_2(\Delta,\Gamma)$ *is contained in the AK regime.)*

b. Suppose that $\Psi_2(\alpha\Delta/(1-\alpha);\Delta) < (\alpha\Delta+\Gamma)/(1-\alpha)$ *(i.e.,* $J_1(\Delta,\Gamma)$ *is contained in the AK regime). Then, if* $X_T \in J_1(\Delta)$ *for some* $T \geq 0$*,* X_t *converges to* $X^{**}(\Delta)$*.*

c. Suppose that $\Psi_2(\alpha(\Delta+\Gamma)/(1-\alpha);\Delta) > \alpha\Delta/(1-\alpha)$ *(i.e.,* $J_2(\Delta,\Gamma)$ *is contained in the AK regime). Then, if* $X_T \in J_2(\Delta,\Gamma)$ *for some* $T \geq 0$*,* X_t *converges to* $X^{**}(\Delta)$*.*

Proof. a. Suppose that both $\Psi(\alpha\Delta/(1-\alpha);\Delta) \geq (\alpha\Delta+\Gamma)/(1-\alpha)$ and $\Psi(\alpha(\Delta+\Gamma)/(1-\alpha);\Delta) \leq \alpha\Delta/(1-\alpha)$ hold. Then, using these inequalities and the fact that Ψ_2 is decreasing, we have $\Phi(\alpha\Delta/(1-\alpha);\Delta) = \Psi_2(\Psi_2(\alpha\Delta/(1-\alpha);\Delta);\Delta) \leq \Psi_2((\alpha\Delta+\Gamma)/(1-\alpha);\Delta) \leq \alpha\Delta/(1-\alpha)$. However, since $\alpha\Delta/(1-\alpha) < X^{**}(\Delta)$, this contradicts the property that $\Phi(X;\Delta) > X$ for all $X < X^{**}(\Delta)$.

b. Since $J_1(\Delta,\Gamma)$ is contained in the AK regime, $\Psi(X;\Delta) = \Psi_2(X;\Delta)$ for all $X \in J_1(\Delta)$. Recall that the unique steady state of mapping Φ, i.e., $X^{**}(\Delta)$, is an element of $[\alpha\Delta/(1-\alpha),(\alpha\Delta+\Gamma)/(1-\alpha)]$ from $\Delta \leq \overline{\Delta}$ and $\Gamma \geq \overline{\Gamma}(\Delta)$.[32] From this and the stability of function Φ as proven in Lemma 1, we have $\Psi_2(\Psi_2(\alpha\Delta/(1-\alpha);\Delta)) = \Phi(\alpha\Delta/(1-\alpha);\Delta) \geq \alpha\Delta/(1-\alpha)$. Since Ψ_2 is monotonic, this means whenever X_T is contained in $J_1(\Delta)$, X_{T+1} is also contained in $J_1(\Delta)$. By iterating this argument, it is shown that X_t stays within $J_1(\Delta)$ for all $t \geq T$ and thus converges to $X^{**}(\Delta)$ by Lemma 1.

c. Since $J_2(\Delta,\Gamma)$ is contained in the AK regime, $\Psi(X;\Delta) = \Psi_2(X;\Delta)$ for all $X \in J_2(\Delta,\Gamma)$. Then, we can show $X_t \in J_2(\Delta,\Gamma)$ for all $t \geq T$ in the same way as above using $\Psi_2(\Psi_2((\alpha\Delta+\Gamma)/(1-\alpha);\Delta);\Delta) = \Phi((\alpha\Delta+\Gamma)/(1-\alpha);\Delta) \leq (\alpha\Delta+\Gamma)/(1-\alpha)$. □

We can also show that the economy cannot stay in the Solow regime or in the modified

[32]See the definitions of $\overline{\Delta}$ and $\overline{\Gamma}$ in (25) and (26).

Solow regime forever without converging to $X^{**}(\Delta)$, unless X_t exceed $X_S^{***}(\Delta, \Gamma)$ at some point in time.[33]

Lemma 3. *Suppose that $\Delta \leq \overline{\Delta}$ and $\Gamma \geq \overline{\Gamma}(\Delta)$. Define $\overline{I}_3$ by*

$$\overline{I}_3 = \begin{cases} X_S^{***}(\Delta, \Gamma) & \text{if } \Delta < \widehat{\Delta} \text{ and } \Gamma \in [\overline{\Gamma}(\Delta), \widehat{\Gamma}(\Delta)], \\ \infty & \text{otherwise,} \end{cases} \tag{42}$$

and intervals I_1, I_2 and I_3, respectively, by

$$I_1 = \left(0, \frac{\alpha\Delta}{1-\alpha}\right], I_2 = \left(\frac{\alpha\Delta}{1-\alpha}, \frac{\alpha\Delta + \Gamma}{1-\alpha}\right), I_3 = \left[\frac{\alpha\Delta + \Gamma}{1-\alpha}, \overline{I}_3\right). \tag{43}$$

Then,

a. If $\Delta = \overline{\Delta}$ and $X_T \in I_1$ for some $T \geq 1$, X_t converges to $X^{**}(\Delta)$. Otherwise, X_t cannot stay within I_1 forever.

b. If $\Gamma = \overline{\Gamma}(\Delta)$ and $X_T \in I_3$ for some $T \geq 1$, X_t converges to $X^{**}(\Delta)$. Otherwise, X_t cannot stay within I_3 forever.

Proof. a. Note that $\Psi(X; \Delta) \geq X$ holds for all $X \in I_1$, where the equality holds only when $\Delta = \overline{\Delta}$ and X is at the right end of I_1 . Since the right end of I_1 is closed and Ψ is continuous, X_t gets out of I_1 in finite periods if $\Delta < \overline{\Delta}$. If $\Delta = \overline{\Delta}$, X_t converges to the upper end of I_1, that is, $\alpha\overline{\Delta}/(1-\alpha)$, which coincides with $X^{**}(\Delta)$ in this case since $\overline{\Gamma}(\overline{\Delta}) = 0$.

b. Note that $\Psi(X; \Delta) \leq X$ holds for all $X \in I_3$, where the equality holds only when $\Gamma = \overline{\Gamma}(\Delta)$ and X is at the left end of I_3. Since the left end of I_3 is closed and Ψ is continuous, X_t gets out of I_3 in finite periods if $\Gamma > \overline{\Gamma}(\Delta)$. If $\Gamma = \overline{\Gamma}(\Delta)$, X_t converges to the lower end of I_3, that is, $(\alpha\Delta + \overline{\Gamma}(\Delta))/(1-\alpha)$, which is $X^{**}(\Delta)$. □

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[33] $X_S^{***}(\Delta, \Gamma)$ and $X_L^{***}(\Delta, \Gamma)$ are respectively the smaller and the larger roots of equation $\Psi_3(X; \Delta, \Gamma) = X$.

Combining these three lemmas, we can prove that X_t will converge to $X^{**}(\Delta)$ as long as it does not exceed $\overline{I}_3$.

Lemma 4. *Suppose that $\Delta \leq \overline{\Delta}$ and $\Gamma \geq \overline{\Gamma}(\Delta)$, and that there is some $T \geq 0$ such that $X_t < \overline{I}_3$ for all $t \geq T$. Then, X_t converges to $X^{**}(\Delta)$.*

Proof. We prove this lemma by reduction to absurdity. Suppose that $X_t < \overline{I}_3$ for all $t \geq T$ and that X_t does not converges to $X^{**}(\Delta)$. Then, the above three lemmas say that X_t cannot stay either within I_1, I_2 or I_3 forever. When $\Psi_2(\alpha\Delta/(1-\alpha);\Delta) < (\alpha\Delta+\Gamma)/(1-\alpha)$, X_t cannot move from $I_1 \cup I_2$ to I_3. This implies that there exist some $T' \geq T$ such that $X_{T'} \in I_1$ and $X_{T'+1} \in I_2$. This means $X_{T'+1} \in J_1(\Delta)$ and hence X_t will converge to $X^{**}(\Delta)$, a contradiction. When $\Psi_2(\alpha\Delta/(1-\alpha);\Delta) \geq (\alpha\Delta+\Gamma)/(1-\alpha)$, Lemma 2 shows that $\Psi_2((\alpha\Delta+\Gamma)/(1-\alpha);\Delta) > \alpha\Delta/(1-\alpha)$ must hold. In this case, X_t cannot move from $I_2 \cup I_3$ to I_1, which implies there exist some $T'' \geq T$ such that $X_{T''} \in I_3$ and $X_{T''+1} \in I_2$. This means $X_{T''+1} \in J_2(\Delta, \Gamma)$ and hence X_t will converge to $X^{**}(\Delta)$, anther contradiction. □

Now, we are ready to prove Propositions 2 and 3.

Proof of Proposition 2. When either '$\Delta < \widehat{\Delta}$ and $\Gamma > \widehat{\Gamma}(\Delta)$' or '$\Delta \in [\widehat{\Delta}, \overline{\Delta}]$ and $\Gamma \geq \overline{\Gamma}(\Delta)$' hold, $\overline{I}_3$ becomes ∞. Thus, $X_t < \overline{I}_3$ holds for all $t \geq 0$, which means X_t converges to $X^{**}(\Delta)$ by Lemma 4. □

Proof of Proposition 3. Let us start from the most obvious claim.

c. This can be easily confirmed from the graph of Ψ_3.

a. If $X_t < X_S^{***}(\Delta, \Gamma)$ holds for all $t \geq 1$, then X_t converges to $X^{**}(\Delta)$ from Lemma 4. Consider the remaining case, where some $T \geq 0$ exist such that $X_T \geq X_S^{***}(\Delta, \Gamma)$. If $X_T > X_S^{***}(\Delta, \Gamma)$, then X_t converges to $X_L^{***}(\Delta, \Gamma)$ by claim *c.* of this proposition. If X_T exactly coincides with $X_S^{***}(\Delta, \Gamma)$, then X_t stays there forever.

b. If $X_S^{***}(\Delta,\Gamma) = X^{**}(\Delta)$, the result directly follows from claim *a.* Otherwise, it is obvious from the graph of Ψ that X_t cannot stay in the neighborhood of $X_S^{***}(\Delta,\Gamma)$ unless there exist some $T \geq 0$ such that $X_T = X_S^{***}(\Delta,\Gamma)$. Suppose that such T exists. Then, X_{T-1} must be an element in $\mathcal{S}_1 \equiv \{X|\Psi(X;\Delta,\Gamma) = X_S^{***}(\Delta,\Gamma)\}$. Note that the number of elements in $\mathcal{S}_1$, denoted by $\sharp\mathcal{S}_1$, does not exceed three since Ψ_1, Ψ_2 and Ψ_3 are all monotonically sloping and thus each curve takes a certain value at most once. In the similar way, we can show that X_t must be an element in $\mathcal{S}_{T-t}$ for all $t < T$, where $\mathcal{S}_\tau$ is defined recursively by $\mathcal{S}_{\tau+1} = \{X|\Psi(X;\Delta,\Gamma) \in \mathcal{S}_\tau\}$ for all $\tau > 0$. By the same reasoning as above, $\sharp\mathcal{S}_{\tau+1}$ does not exceed $3\mathcal{S}_\tau$, which implies $\sharp\mathcal{S}_\tau \leq 3^\tau$. Also note that $\mathcal{S}_{\tau+1} \supset \mathcal{S}_\tau$ holds since S_τ always contains a fixed point, $X_S^{***}(\Delta,\Gamma)$. From these arguments, if X_t converges to $X_S^{***}(\Delta,\Gamma)$, X_0 must be an element of $\mathcal{S}_{T-1}$ for some $T \geq 1$, that is, $X_0 \in \bigcup_{\tau=0}^{\infty} \mathcal{S}_\tau = \mathcal{S}_\infty$. Since $\sharp\mathcal{S}_\tau \leq 3^\tau$ for all $\tau > 0$, $\sharp\mathcal{S}_\infty$ is countable. This shows the set of initial states from which X_t converges to $X_S^{***}(\Delta,\Gamma)$ has measure 0. Combining this result with claim *a*, we are done.

d. We are considering the case of $\Gamma > \overline{\Gamma}(\Delta)$ since $\Gamma = \overline{\Gamma}(\Delta)$ implies $X^{**}(\Delta) = X_S^{***}(\Delta,\Gamma)$ which we have ruled out. Note also that $\overline{\Gamma}(\Delta) > 0$ since we are assuming that $0 < \Delta \leq \widehat{\Delta} < \overline{\Delta}$. Thus $0 < \overline{\Gamma}(\Delta) < \Gamma$ holds, which implies that $X^{**}(\Delta)$ is located in the interior of $J_1(\Delta) \cap J_2(\Delta,\Gamma)$. From lemma 2, X_t converges to $X^{**}(\Delta)$ whenever X_t enters $J_1(\Delta) \cap J_2(\Delta,\Gamma)$, which means that $X^{**}(\Delta)$ is locally stable. $\square$